Concursul Național Studențesc de Matematică "Traian Lalescu"

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BAREM
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Problema 1.

Fie $f : \mathbb{C} \to \mathbb{C}$ olomorfă pe $\mathbb{C}$, pentru care $f(0) = 0$ şi

$$\mathrm{Re}\,(f) = x \cdot \varphi(y) + y \cdot \psi(x), \quad \forall (x, y) \in \mathbb{R}^2,\ \varphi, \psi \in C^2(\mathbb{R}),\ z = x + iy.$$

a) Determinaţi funcţia $f(z)$;

b) Arătaţi că $\displaystyle\int_0^\infty e^{ix^2} dx = \frac{\sqrt{2\pi}}{4}(1 + i)$;

c) Dacă $f(1) = \frac{i}{24}$, $f'(0) = f''(0) = 0$ rezolvaţi în $\mathbb{C}$ ecuaţia

$$f(z) = \frac{1}{6\pi}\left(\int_0^\infty e^{ix^2}\,\mathrm{d}x\right)^2.$$

Rezolvare:

a) $\Delta u = 0 \Rightarrow y \cdot \psi''(x) + x \cdot \varphi''(y) = 0$

$\dfrac{\psi''(x)}{x} = -\dfrac{\varphi''(y)}{y} = C =$ constantă **2,5p**

Aflarea $\psi(x) = \dfrac{Cx^3}{6} + a_1 x + a_2,\ a_1, a_2 \in \mathbb{R}$ **1p**

Aflarea $\varphi(y) = -\dfrac{Cy^3}{6} + b_1 y + b_2,\ b_1, b_2 \in \mathbb{R}$ **1p**

$f(z) = \alpha z - \dfrac{i\beta z^2}{2} - \dfrac{iCz^4}{24} + d,\ \alpha, \beta, d$ constante **1p**

$f(0) = 0 \Rightarrow d = 0$ **0,5p**

b) $\displaystyle\int_0^\infty e^{ix^2}\,\mathrm{d}x \overset{x^2=t}{=} \frac{1}{2}\int_0^\infty \frac{1}{\sqrt{t}} e^{it}\,\mathrm{d}t$

$= \dfrac{1}{2} \cdot \mathcal{L}\left[\dfrac{1}{\sqrt{t}}\right](-i) = \dfrac{1}{2}\sqrt{\pi} \cdot \sqrt{i}$

pe ramura principală $\sqrt{i} = \dfrac{\sqrt{2}}{2}(1 + i)$ **2p**

c) Impunerea condiţiilor cerute conduce la

$\alpha = 0,\ \beta = 0,\ C = -1$ şi $f(z) = \dfrac{iz^4}{24}$ **1p**

Rezolvarea ecuaţiei $z^4 = 1 \Rightarrow z \in \{\pm 1, \pm i\}$ **1p**

Problema 2.

Calculaţi

$$\int_{|z|=3} (1+z^2)\left(e^{\frac{1}{z-1}} + e^{\frac{1}{z-2}}\right)\, \mathrm{d}z.$$

Rezolvare:

$I_1 = \displaystyle\int_{|z|=3} f_1(z)\, \mathrm{d}z,\ f_1(z) = (1+z^2)e^{\frac{1}{z-1}}$ **0,5p**

$I_2 = \displaystyle\int_{|z|=3} f_2(z)\, \mathrm{d}z,\ f_2(z) = (1+z^2)e^{\frac{1}{z-2}}$ **0,5p**

Calculul lui I_1:

$z = 1$ este punct singular esenţial **0,5p**

$\mathrm{Rez}(f_1(z), 1) =$ coeficientul c_{-1} din dezvoltarea

$\left[(z-1)^2 + 2(z-1) + 2\right] \cdot \displaystyle\sum_{k=0}^{\infty} \frac{1}{(z-1)^k \cdot k!} = \frac{19}{6}$ **3p**

$I_1 = \dfrac{19\pi i}{3}$ **0,5p**

Calculul lui I_2:

$z = 2$ este punct singular esenţial **0,5p**

$\mathrm{Rez}(f_2(z), 2) =$ coeficientul c_{-1} din dezvoltarea

$\left[(z-2)^2 + 4(z-2) + 5\right] \cdot \displaystyle\sum_{k=0}^{\infty} \frac{1}{(z-2)^k \cdot k!} = \frac{43}{6}$ **3p**

$I_1 = \dfrac{43\pi i}{3}$ **0,5p**

$I = I_1 + I_2 = \dfrac{62\pi i}{3}$ **1p**

Problema 3.

Se consideră funcţiile $f, g : \mathbb{R} \to \mathbb{R}$, $f(x) = \dfrac{1}{x^2+1}$ şi $g(x) = (1-x^2) \cdot f^2(x)$ şi fie

$$h_n(x) = \underbrace{f(x) * f(x) * ... * f(x)}_{n \text{ ori}},\ n \in \mathbb{N},\ n \geq 2,$$

unde $*$ reprezintă produsul de convoluţie.

a) Calculaţi transformatele Fourier $\hat{f}(\omega)$ şi $\hat{g}(\omega)$.

b) Calculaţi $\displaystyle\int_{-\infty}^{\infty} h_{2026}(x) \cdot \sin x \cdot \sin(2x)\, \mathrm{d}x$.

Rezolvare:

a) Calculează $\hat{f}(\omega) = \begin{cases} \pi e^{\omega},\ \omega < 0, \\ \pi e^{-\omega},\ \omega > 0, \\ \pi,\ \omega = 0 \end{cases}$ sau $\hat{f}(\omega) = \pi \cdot e^{-|\omega|}$ **2p**

Scrie $g(x) = \big(x \cdot f(x)\big)'$ şi obţine $\hat{g}(\omega) = \pi \cdot |\omega| \cdot e^{-|\omega|}$ **3p**

b) $I = \dfrac{1}{2}\displaystyle\int_{-\infty}^{\infty} h_{2026}(x) \cdot \cos(x)\, \mathrm{d}x - \dfrac{1}{2}\int_{-\infty}^{\infty} h_{2026}(x) \cdot \cos(3x)\, \mathrm{d}x$ **1p**

$I = \dfrac{1}{2}\mathrm{Re}(\hat{h}_{2026}(1)) - \dfrac{1}{2}\mathrm{Re}(\hat{h}_{2026}(3))$ **2p**

$\hat{h}_n(\omega) = \big(\hat{f}(\omega)\big)^n$ **1p**

Finalizare $I = \dfrac{\pi^{2026}}{2}\big(e^{-2026} - e^{-6078}\big)$ **1p**

Problema 4.

Să se rezolve în clasa funcţiilor original Laplace problema:

$$x''(t) + 2x'(t) + 2x(t) + \int_0^t (t-\tau)e^{-(t-\tau)}x(\tau)\,\mathrm{d}\tau = 0,\ t \geq 0$$

$$x(0) = 0,\ x'(0) = -1.$$

Rezolvare:

Aplică transformarea Laplace, recunoaşte produsul de convoluţie şi obţine:
$\mathcal{L}[x(t)](s) \cdot \left(s^2 + 2s + 2 + \dfrac{1}{(s+1)^2}\right) = -1$ **3p**
Scrie $\mathcal{L}[x(t)](s) = \dfrac{-(s+1)^2}{(s+1)^4 + (s+1)^2 + 1} = \mathcal{L}[e^{-t}f(t)](s)$, unde $f(t) = \mathcal{L}^{-1}\left[\dfrac{-s^2}{s^4 + s^2 + 1}\right](t)$ .. **3p**
Desface în fracţii simple: $\dfrac{-s^2}{s^4+s^2+1} = -\dfrac{1}{2}\dfrac{s}{s^2-s+1} + \dfrac{1}{2}\dfrac{s}{s^2+s+1}$ **1p**
Găseşte
$f(t) = -\dfrac{1}{2}e^{\frac{1}{2}t}\cos\left(\dfrac{\sqrt{3}}{2}t\right) - \dfrac{1}{2\sqrt{3}}e^{\frac{1}{2}t}\sin\left(\dfrac{\sqrt{3}}{2}t\right) + \dfrac{1}{2}e^{-\frac{1}{2}t}\cos\left(\dfrac{\sqrt{3}}{2}t\right) - \dfrac{1}{2\sqrt{3}}e^{-\frac{1}{2}t}\sin\left(\dfrac{\sqrt{3}}{2}t\right)$ .. **2p**
Găseşte
$x(t) = -e^{-t}\left[\mathrm{sh}\left(\dfrac{1}{2}t\right)\cos\left(\dfrac{\sqrt{3}}{2}t\right) + \dfrac{1}{\sqrt{3}}\mathrm{ch}\left(\dfrac{1}{2}t\right)\sin\left(\dfrac{\sqrt{3}}{2}t\right)\right],\ t \geq 0$ **1p**